Enrollment No: ____________________ **Exam Seat No: ____________________**

C. U. SHAH UNIVERSITY

Summer Examination-2022

Subject Name : Functional Analysis

Subject Code : 5SC03FUA1 **Branch: M.Sc. (Mathematics)**

Semester: 3 **Date: 25/04/2022** **Time: 02:30 To 05:30** **Marks: 70**

Instructions:

(1) Use of Programmable calculator and any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

SECTION – I

Q-1 **Attempt the Following questions** **(07)**

a. The closure of a convex set in a normed space is a _________.

b. State Jensen's inequality.

c. Let Y be a closed subspace of a normed space X. If x and y are in X then, $|||x + y + Y||| \leq |||x + Y||| + |||y + Y|||$. True or False.

d. Let X and Y be normed linear spaces. If X is finite dimensional then every linear map from X to Y is continuous. True or False.

e. If X is a metric space, then both $\emptyset$ and X are _______ in X

f. If X be a norm linear space then X' is always complete. True or False.

g. Define : Convex set

Q-2 **Attempt all questions** **(14)**

a. Let $\{a_j\}$ and $\{b_j\}$ be sequence in K. Let $1 < p < \infty$ and $q \in R$ such that $\mathrm{p} + \mathrm{q} = \mathrm{pq}$. Then show that $\sum_{j=1}^{n} |a_j b_j| \leq \left(\sum_{j=1}^{n} |a_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |b_j|^q\right)^{\frac{1}{q}}$. State the result you use. **(06)**

b. Let $\|\cdot\|$ and $\|\cdot\|_0$ be norms on a linear space X. When is $\|\cdot\|$ stronger than $\|\cdot\|_0$? Prove that $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent if and only if there are positive constants α and β such that $\alpha\|\cdot\| \leq \|\cdot\|_0 \leq \beta\|\cdot\|$. **(06)**

c. State Minkowski's inequality. **(02)**

OR

Q-2 **Attempt all questions** **(14)**

a. Let $r, p \in [1, \infty]$ with $r < p$. Then prove that $l^p \subset l^r$ and $\|x\|_r \leq \|x\|_p \quad \forall\, x \in l^p$. **(06)**

b. Let $l^p = \left\{x = \{x_j\}_{j=1}^{\infty} \; and \; \sum_{j=1}^{\infty} |x_j|^p < \infty\right\}$, **(06)**

consider $\|x\|_p = \left(\sum_{j=1}^{\infty}|x_j|^p\right)^{\frac{1}{p}}$. Then show that $(l^p, \|\cdot\|_p)$ is a normed space.

c. Let Y be a closed subspace of a normed space X. If x and y are in X then show that $|||x+y+Y||| \le |||x+Y||| + |||y+Y|||$. **(02)**

Q-3 **Attempt all questions** **(14)**

a. Define transpose of bounded linear function. Let X, Y be two norm linear spaces and $F \in BL(X,Y)$ then prove that $F' \in BL(Y', X')$. **(07)**

b. State and prove Hahn-Banach extension theorem. **(07)**

OR

Q-3 **a.** Let X, Y be two norm linear space and $F: X \to Y$ be a linear map. Then prove the following are equivalent. **(08)**

i. F is bounded on $\bar{U}(0,r)$ for some $r > 0$.
ii. F is continuous at 0.
iii. F is continuous on X.
iv. F is uniformly continuous on X.
v. $\|F(x)\| \le \alpha \|x\|$ for all $x \in X$ and some $\alpha > 0$.

b. State and prove Holder's inequality. **(06)**

SECTION – II

Q-4 **Attempt the Following questions** **(07)**

a. State Schur's Lemma.
b. Define spectrum of an operator.
c. State bounded inverse theorem.
d. In which case we can get always unique Hahn – Banach extension?
e. Let X be a norm linear space over K. Then every non zero linear functional f on X is an open map. True/ False.
f. Define: Absolutely summable series.
g. Define: Dual of a norm linear space.

Q-5 **Attempt all questions** **(14)**

a. State and prove closed graph theorem. **(10)**

b. Let X and Y be Banach spaces. Show that the product space $X \times Y$, with the norm defined by $\|(x, y)\| = \|x\| + \|y\|$, $(x,y) \in X \times Y$, is Banach space. **(04)**

OR

Q-5 **a.** State and prove uniform boundedness principal. **(07)**

b. Let X be a separable normed space. Then show that every bounded sequence in X' has a weak* convergent subsequence. **(07)**

Q-6 **Attempt all questions** **(14)**

a. Let X be norm linear space then prove that following are equivalent: **(09)**

i) X is a Banach space.
ii) Every absolutely summable series in X is summable.

b. Prove that every normed linear space is isometrically isomorphic to dense subspace of Banach space. **(05)**

OR

Q-6 **Attempt all Questions**

a. Let X be a normed space. Define spectrum, eigen spectrum and approximate eigen spectrum of $A \in BL(X)$. If A is of finite rank, then show that $\sigma_e(A) = \sigma_a(A) = \sigma(A)$. **(09)**

b. Let Z be a closed subspace of a normed space X. Let $Q: X \to X/Z$ be $Q(x) = x + Z$. Show that Q is continuous and open. State the result you use. **(05)**